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STRESS STATE OF AN ELASTIC SPACE WITH A TOROIDAL-SHAPED CAVITY

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Solutions are obtained for the stress state problem for an elastic space with an internal toroidal-shaped cavity that can be expanded in a trigonometric series in the angle in cylindrical coordinates. Displacements and stresses are specified on the boundary. An analytic solution of the problem is found using generalized analytic functions. Stresses and displacements of points in the elastic space are calculated.

Key words: *elastic space, toroidal-shaped cavity, generalized analytic functions.*

Introduction. Methods for the solution of plane elasticity problems using classical analytic functions of a complex variable were developed by Muskhelishvili [1]. Methods extending the apparatus of analytic and generalized analytic functions to dimensional problems are described in [2]. Using these methods to find problem solutions that admit an expansion in a trigonometric series in the angle in cylindrical coordinates, one can obtain an analytic solution of some canonical problems. Among such problems is the stress state problem of an elastic body in the shape of a solid or hollow torus or a space having a toroidal-shaped cavity.

1. Generalized Analytic Functions. Following [3], we shall call a generalized analytic function (GAF) a continuously differentiable complex-valued function Φ of the variables z and r that satisfies the equation

$$\Big(\frac{\partial}{\partial z} + i\,\frac{\partial}{\partial r}\Big)\Phi - \frac{2m+1}{t-\bar t}\,(\Phi - \bar\Phi) = 0, \tag{1}$$

where m is an integer and $t = z + ir$. We denote such GAFs by $\Phi_m(t)$. The properties of the functions satisfying Eq. (1) were studied in [4] and other papers.

Separating the real and imaginary terms in (1), we obtain the equalities

$$\frac{\partial}{\partial z}\,\mathrm{Re}\,\Phi_m = \frac{\partial}{\partial r}\,\mathrm{Im}\,\Phi_m + \frac{2m+1}{r}\,\mathrm{Im}\,\Phi_m,$$

$$\frac{\partial}{\partial r}\,\mathrm{Re}\,\Phi_m = -\frac{\partial}{\partial z}\,\mathrm{Im}\,\Phi_m, \tag{2}$$

which are similar to the Cauchy–Riemann conditions for classical analytic functions. If the real and imaginary terms of a GAF are doubly differentiable functions, equalities (2) lead to

$$\nabla^2_m(r^m\,\mathrm{Re}\,\Phi_m) = 0, \qquad \nabla^2_{m+1}(r^m\,\mathrm{Im}\,\Phi_m) = 0 \qquad \Big(\nabla^2_m = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} - \frac{m^2}{r^2}\Big).$$

Following [5], we introduce the derivative of the function Φ_m:

$$\Phi'_m(t) = \frac{\partial\Phi_m}{\partial z} = -i\,\frac{\partial\Phi_m}{\partial z} + \frac{2m+1}{r}\,\mathrm{Im}\,\Phi_m. \tag{3}$$

If the function Φ_m is doubly differentiable with respect to z and r, Φ'_m satisfies Eq. (1). (It should be noted that the sum or difference of GAFs and their product into a real constant are GAFs satisfying Eq. (1), but multiplication by an imaginary constant removes the function from the class of functions considered.)

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The functions $\Phi_m(t)$ can be treated as pseudoanalytic Bers functions with a generating pair $F = 1$, $G = ir^{-2m-1}$. In [5], the zeroes and singular points are classified for $\operatorname{Im}(\bar{F}G) > 0$ and analogs of the Cauchy theorem and formula are constructed.

2. Representation of the General Elasticity Solution in Cylindrical Coordinates Using Generalized Analytic Functions. Let the elastic displacement components in cylindrical coordinates z, r, and θ be expanded in trigonometric series:

$$w(z,r,\theta) = \sum_{n=0}^{\infty}(w_n^1(z,r)\sin n\theta + w_n^2(z,r)\cos n\theta),$$

$$u(z,r,\theta) = \sum_{n=0}^{\infty}(u_n^1(z,r)\sin n\theta + u_n^2(z,r)\cos n\theta), \tag{4}$$

$$v(z,r,\theta) = \sum_{n=0}^{\infty}(v_n^1(z,r)\cos n\theta - v_n^2(z,r)\sin n\theta).$$

The expansion coefficients should satisfy the following equations (below, the superscripts are omitted):

$$\nabla_n^2 w_n + \frac{1}{1-2\nu}\,\frac{\partial\vartheta_n}{\partial z} = 0,$$

$$\nabla_{n+1}^2(u_n - v_n) + \frac{1}{1-2\nu}\Big(\frac{\partial\vartheta_n}{\partial r} - \frac{n}{2}\,\vartheta_n\Big) = 0,$$

$$\nabla_{n-1}^2(u_n + v_n) + \frac{1}{1-2\nu}\Big(\frac{\partial\vartheta_n}{\partial r} + \frac{n}{2}\,\vartheta_n\Big) = 0.$$

Here $\vartheta_n = \partial w_n/\partial z + \partial u_n/\partial r + (u_n - nv_n)/r$.

In [6–9], the following expressions for the expansion coefficients w_n, u_n, and v_n are obtained:

$$2G[w_n + i(u_n - s_m v_n)] = r^m(æ\Phi_{m1} - 2z\bar{\Phi}'_{m1} - \bar{\Phi}_{m2} - 2is_m \operatorname{Im}\Phi_{m3}). \tag{5}$$

Here $s_m = 1$ for $m > 0$; $s_m = 0$ for $m = 0$; $s_m = -1$ for $m < 0$; $m = \pm n$; $æ = 3 - 4\nu$; ν is Poisson's ratio; G is the shear modulus; $\Phi_{mj}(t)$ are generalized analytic functions of the complex variable $t = z + ir$ that satisfy Eq. (1) and the equality

$$r^m \operatorname{Re}\Phi'_{mj} = r^{-m}\operatorname{Re}\Phi'_{-mj} \qquad (j = 1,2,3). \tag{6}$$

The stress components are also expanded in series of the form (3). For the expansion coefficients $\sigma_{zn}^{1,2}(z,r)$, $\sigma_{rn}^{1,2}(z,r)$, $\sigma_{\theta n}^{1,2}(z,r)$, $\tau_{zrn}^{1,2}(z,r)$, $\tau_{z\theta n}^{1,2}(z,r)$, and $\tau_{r\theta n}^{1,2}(z,r)$, the following expressions are obtained:

$$\sigma_{zn} + i(\tau_{zrn} - s_m\tau_{z\theta n}) = r^m(\Phi'_{m1} - 2z\bar{\Phi}''_{m1} - \bar{\Phi}'_{m2} - is_m\operatorname{Im}\Phi'_{m3}),$$

$$\sigma_{zn} + \sigma_{rn} + \sigma_{\theta n} = 4(1+\nu)r^m\operatorname{Re}\Phi'_{m1}, \qquad \sigma_{\theta n} = 2\nu r^m\operatorname{Re}\Phi'_{m1} + 2G(u_n - nv_n)/r,$$

$$\tau_{r\theta n} = r^m\operatorname{Re}\Phi'_{m3} + 2G(nu_n - v_n)/r.$$

3. Generalized Analytic Functions in Toroidal Coordinates. We shall consider problems in toroidal coordinates (ξ,η,θ) (Fig. 1). Here ξ and η are bipolar coordinates in the meridional plane $\theta = \text{const}$ which are related to cylindrical coordinates by the formulas

$$z = c\,\frac{\sin\eta}{\cosh\xi - \cos\eta}, \qquad r = c\,\frac{\sinh\xi}{\cosh\xi - \cos\eta}$$

(c is a parameter of the coordinate system).

Following [8], we seek generalized analytic functions Φ_{mj} in the form of the series

$$\Phi_{mj} = \frac{2\sqrt{\pi}}{\Gamma(1/2-m)}\,(\alpha_{mj}\Theta_{m,m-1} + \beta_{mj}\Theta_{m,m} + \gamma_{mj}\Theta_{m,m+1}) + \sum_{k=1}^{\infty}(a_{mk}^{js}\Phi_{mk}^{s} + a_{mk}^{jc}\Phi_{mk}^{c}), \tag{7}$$

where α_{mj}, β_{mj}, γ_{mj}, a_{mk}^{js}, and a_{mk}^{jc} are real coefficients, Γ is the gamma-function, and Φ_{mk}^{s} and Φ_{mk}^{c} are generalized analytic functions, which in toroidal coordinates are written as

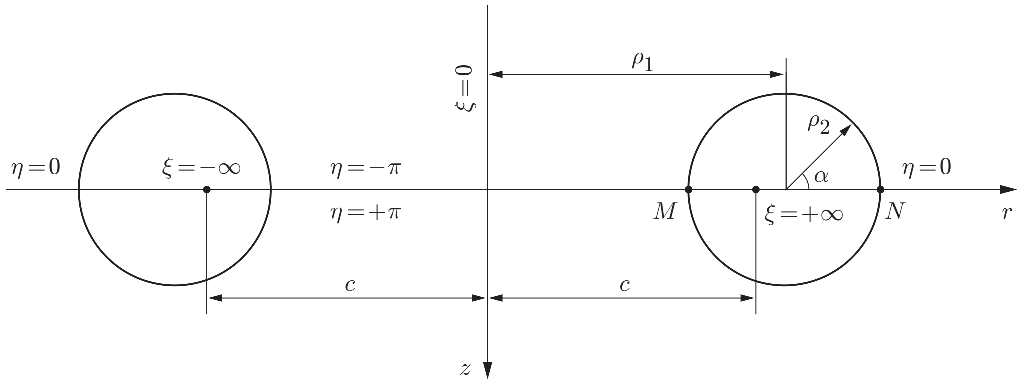

Fig. 1

$$\Phi^{s,c}_{mk} = \frac{1}{r^m}\sqrt{2(\cosh\xi-\cos\eta)}\Big\{(2k-2m-1)P^m_{k-1/2}(\cosh\xi)\binom{\sin k\eta}{\cos k\eta}$$
$$-(2k+2m-1)P^m_{k-3/2}(\cosh\xi)\binom{\sin(k-1)\eta}{\cos(k-1)\eta}$$
$$-2i\Big[P^{m+1}_{k-1/2}(\cosh\xi)-\binom{\cos k\eta}{-\sin k\eta}-P^{m+1}_{k-3/2}(\cosh\xi)\binom{\cos(k-1)\eta}{-\sin(k-1)\eta}\Big]\Big\}\quad (k\geqslant 1), \tag{8}$$
$$\Phi^s_{m0}=-\Phi^s_{m1},\qquad \Phi^c_{m0}=\Phi^c_{m1}.$$

Here $P^m_{k-1/2}$ are first-order associated Legendre functions of half-integer index and Θ_{mq} are multivalued generalized analytic functions [6, 7]. After counterclockwise tracing around the pole $(0,c)$, their increments $\Delta\Theta_{m,m-1} = (-2r-4imz)c^{m-1}/r^{2m+1}$, $\Delta\Theta_{m,m}=-2ic^m/r^{2m+1}$, and $\Delta\Theta_{m,m+1}=-2/c^{m+1}$. According to [7], for these functions, the following relations hold:

$$r^m\operatorname{Re}\Theta_{m,m}=\frac{\Gamma(1/2-m)}{2\sqrt{\pi}\,c}\sqrt{2(\cosh\xi-\cos\eta)}\,P^m_{-1/2}(\cosh\xi),\qquad \operatorname{Re}\Theta_{m,m-1}=\frac{1}{r}\operatorname{Im}\Theta_{m-1,m-1},$$
$$\operatorname{Im}\Theta_{m,m-1}=2m\frac{z}{r}\operatorname{Im}\Theta_{m,m}+\frac{\Gamma(1/2-m)}{2\sqrt{\pi}\,cr^m}\frac{1}{\sqrt{2(\cosh\xi-\cos\eta)}}$$
$$\times(4m\sinh\xi P^m_{-1/2}(\cosh\xi)+4(\cosh\xi-\cos\eta)P^{m+1}_{-1/2}(\cosh\xi)), \tag{9}$$
$$\operatorname{Re}\Theta_{m,m+1}=r^{-2m-1}\operatorname{Im}\Theta_{-m-1,m+1},$$
$$\operatorname{Im}\Theta_{m,m+1}=\frac{\Gamma(1/2-m)}{2\sqrt{\pi}\,cr^m}\sqrt{2(\cosh\xi-\cos\eta)}\,\frac{2}{1+2m}P^{m+1}_{-1/2}(\cosh\xi).$$

According to (3), the derivatives of functions (8) are calculated from the formulas

$$\frac{\partial}{\partial z}\Phi^{s,c}_{mk}=\pm\frac{1}{4c}[(2k-2m-1)\Phi^{c,s}_{m,k+1}-(4k-2)\Phi^{c,s}_{mk}+(2k+2m-1)\Phi^{c,s}_{m,k-1}],$$
$$\Theta'_{m,m-1}=\frac{\Gamma(1/2-m)}{4\sqrt{\pi}c^2}\Phi^c_{m1}+\frac{2m}{c}\Theta_{m,m}, \tag{10}$$
$$\Theta'_{m,m}=-\frac{\Gamma(1/2-m)}{4\sqrt{\pi}c^2}\Phi^s_{m1},\qquad \Theta'_{m,m+1}=\frac{\Gamma(1/2-m)}{4\sqrt{\pi}c^2}\Phi^c_{m1}.$$

From the condition that the displacements are continuous and unique, we obtain

$$\alpha_{m2}=æ\alpha_{m1},\qquad (æ+1)\alpha_{m1}=s_m\alpha_{m3},\qquad \gamma_{m2}=æ\gamma_{m1},$$
$$(æ+1)\gamma_{m1}=-s_m\gamma_{m3},\qquad æ\beta_{m1}+\beta_{m2}=2s_m\beta_{m3}\qquad (m\neq 0), \tag{11}$$
$$\alpha_{01}=\alpha_{02}=\alpha_{03}=0,\qquad \gamma_{02}=æ\gamma_{01},\qquad \beta_{02}=-æ\beta_{01},\qquad \beta_{03}=0.$$

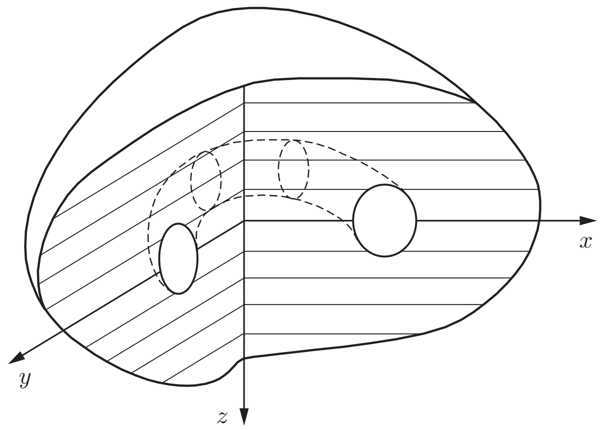

Fig. 2

4. Solution of Problems for an Elastic Space Having a Toroidal-Shaped Cavity with Displacements Specified on the Cavity Surface. We consider an elastic space with an internal cavity in the shape of a circumferential torus (Fig. 2). We assume that the axis of the torus coincides with the Oz axis, the geometrical place of the centers of the sections cut by the planes $\theta = \text{const}$ is a circumference of radius ρ_1 in the plane $z = 0$, and ρ_2 is the cross-sectional radius of the torus. The equation of the cavity surface in toroidal coordinates has the form $\xi = \xi_0 = \ln [(\rho_1 + c)/\rho_2]$, where $c = \sqrt{\rho_1^2 - \rho_2^2}$.

Let displacements on the surface of the torus be defined by the trigonometric series (4). Then, the expansion coefficients can be written as

$$2Gw_n = \frac{1}{\sqrt{\lambda - 2\cos\eta}} \sum_{k=0}^{\infty} (A_k^s \sin k\eta + A_k^c \cos k\eta),$$

$$2G(u_n - v_n) = -\frac{2}{\sqrt{\lambda - 2\cos\eta}} \sum_{k=0}^{\infty} (B_k^s \cos k\eta - B_k^c \sin k\eta), \tag{12}$$

$$2G(u_n + v_n) = -\frac{2}{\sqrt{\lambda - 2\cos\eta}} \sum_{k=0}^{\infty} (C_k^s \cos k\eta - C_k^c \sin k\eta) \qquad (\lambda = 2\cosh \xi_0).$$

Substituting series (12) into (5) and using expressions (7)–(11), we obtain the following relations for $A_k^{s,c}$ (the superscripts s and c are omitted):

$$A_k = l_k - q_k/2. \tag{13}$$

Here

$$l_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad l_1 = \binom{0}{-2} L_0 + L_1 - L_2, \qquad l_0 = \binom{0}{1} L_0 - \binom{0}{1} L_1,$$

$$L_k = [(2k - 2n - 1)(æa_{nk}^1 - a_{nk}^2) + æ\beta_{n1} - \beta_{n2}] P_{k-1/2}^n(\lambda/2) \qquad (k \geqslant 1),$$

$$L_0 = [-(1 + 2n)(æa_{n1}^1 - a_{n1}^2) + æ\beta_{n1} - \beta_{n2}] P_{-1/2}^n(\lambda/2),$$

$$q_k = E_{k-1} - E_{k+1} \quad (k \geqslant 2), \qquad q_1 = \binom{2}{0} E_0 - E_2, \qquad q_0 = \binom{0}{-1} E_1,$$

$$E_k = [(2k - 2n - 1)d_k^1 - (2k + 2n + 1)d_{k+1}^1] P_{k-1/2}^n(\lambda/2) \qquad (k \geqslant 1),$$

$$E_0 = [8n\alpha_{n1} - (2n + 1)d_1^1] P_{-1/2}^n(\lambda/2).$$

The relations for the coefficients $B_k^{s,c}$ are written as follows (the superscripts s and c are omitted):

$$B_k = l_k + q_k/2. \tag{14}$$

Here

$$l_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad l_1 = \binom{-2}{0} L_0 + L_1 - L_2,$$

$$l_0 = L_0 - L_1 + 4n\alpha_{n1} P^n_{-1/2}(\lambda/2)\sqrt{(\lambda/2)^2 - 1},$$

$$L_k = [(æa^1_{nk} + a^2_{nk} - 2a^3_{nk}) - (æa^1_{n,k+1} + a^2_{n,k+1} - 2a^3_{n,k+1})] P^{n+1}_{k-1/2}(\lambda/2) \qquad (k \geqslant 1),$$

$$L_0 = [-(æa^1_{n1} + a^2_{n1} - 2a^3_{n1}) + 2\alpha_{n1} + (-2æ\gamma_{n1} + 2\gamma_{n3})/(1+2n)] P^{n+1}_{-1/2}(\lambda/2),$$

$$q_k = E_{k-1} - E_{k+1} \quad (k \geqslant 2), \qquad q_1 = \binom{0}{2} E_0 - E_2, \qquad q_0 = -\binom{1}{0} E_1,$$

$$E_k = (d^1_k - d^1_{k+1}) P^{n+1}_{k-1/2}(\lambda/2) \quad (k \geqslant 1), \qquad E_0 = -d^1_1 P^{n+1}_{-1/2}(\lambda/2).$$

In the formulas given above,

$$d^j_k = (2k - 2n - 3) a^j_{n,k-1} - (4k-2) a^j_{nk} + (2k + 2n + 1) a^j_{n,k+1} \qquad (k \geqslant 2),$$

$$d^j_1 = -\binom{1-2n}{3+2n} a^j_{n1} + (3+2n) a^j_{n2} + 2\binom{\alpha_{nj} + \gamma_{nj}}{\beta_{nj}}.$$

For the factors written in matrix form, the upper row of the matrix corresponds to the subscript s and the lower row to the subscript c.

The formulas for the coefficients $C^{s,c}_{nk}$ are obtained from (14) by replacing n by $-n$ for

$$L_k = [(æa^1_{-nk} + a^2_{-nk} + 2a^3_{-nk}) - (æa^1_{-n,k+1} + a^2_{-n,k+1} + 2a^3_{-n,k+1})] P^{-n+1}_{k-1/2}(\lambda/2) \qquad (k \geqslant 1),$$

$$L_0 = [-(æa^1_{-n1} + a^2_{-n1} + 2a^3_{-n1}) + 2\alpha_{-n1} + (-2æ\gamma_{-n1} + 2\gamma_{-n3})/(1-2n)] P^{-n+1}_{-1/2}(\lambda/2).$$

Relations (13) and (14) for specified A_k, B_k, and C_k ($k = 0, 1, 2, \ldots$) and conditions (6) form an infinite system of equations for the coefficients of the series (7). To calculate the stresses and displacements in the elastic space, it is necessary to determine these coefficients.

5. Solution of Problems for an Elastic Space with a Toroidal-Shaped Cavity under External Forces Specified on the Cavity Surface. We consider two cases where external forces are specified on the cavity surface.

1. Let surface stresses be specified in the form

$$2\sigma_{zn} = \frac{1}{4c} \frac{1}{\sqrt{\lambda - 2\cos\eta}} \sum_{k=0}^{\infty} (S^s_k \cos k\eta - S^c_k \sin k\eta),$$

$$\tau_{zrn} - \tau_{z\theta n} = -\frac{2}{4c} \frac{1}{\sqrt{\lambda - 2\cos\eta}} \sum_{k=0}^{\infty} (T^s_k \sin k\eta + T^c_k \cos k\eta). \qquad (15)$$

The expressions for $\sigma_{zn} + \sigma_{rn}$, $\tau_{r\theta n}$, and $(\sigma_{zn} + \sigma_{rn} + \sigma_{\theta n})/(1+\nu)$ have the same form as for σ_{zn} (S is replaced by R, V, and Z, respectively), and the expression for $\tau_{zrn} + \tau_{z\theta n}$ is obtained by replacing T by U in the second formula (15).

The procedure for determining the coefficients in the series (15) is described in Sec. 4. In the first formula (15), we have

$$S_k = l_k - q_k/2,$$

where

$$l_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad l_1 = \binom{-2}{0} L_0 + L_1 - L_2,$$

$$L_k = [(2k - 2n - 1)(d^1_k - d^2_k) - (2k + 2n + 1)(d^1_{k+1} - d^2_{k+2})] P^n_{k-1/2}(\lambda/2) \qquad (k \geqslant 1),$$

$$L_0 = [-(1+2n)(d^1_1 - d^2_1) + 8n(1-æ)\alpha_{n1}] P^n_{-1/2}(\lambda/2),$$

$$R_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad R_1 = \binom{-2}{0} L_0 + L_1 - L_2, \qquad R_0 = L_0 - L_1,$$

$$E_k = [(2k-2n-1)h_k - (2k+2n+1)h_{k+1}]P^n_{k-1/2}(\lambda/2) \quad (k \geqslant 1), \qquad E_0 = -(2n+1)h_1 P^n_{-1/2}(\lambda/2),$$

$$h_k = (2k-2n-3)d^1_{k-1} - (4k-2)d^1_k + (2k+2n+1)d^1_{k+1} \quad (k \geqslant 2),$$

$$h_1 = -\binom{3+2n}{1-2n} d^1_1 + (3+2n)d^1_2 + \binom{16n}{0}\alpha_{n1}.$$

The coefficients R_k take the form

$$R_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad R_1 = \binom{-2}{0} L_0 + L_1 - L_2, \qquad R_0 = L_0 - L_1,$$

where $L_k = Z^1_k + 4((n+1)B_k - (n-1)C_k)/\sqrt{\lambda^2-4}$.

Next, the coefficients V_k have the form

$$V_k = -L_{k-1} + L_k - L_{k+1} \quad (k \geqslant 2), \qquad V_1 = \binom{-2}{0} L_0 + L_1 - L_2, \qquad V_0 = L_0 - L_1,$$

where

$$L_k = Z^3_k - 4((n+1)B_k + (n-1)C_k)/\sqrt{\lambda^2-4},$$

$$Z^j_k = [(2k-2n-1)d^j_k - (2k+2n+1)d^j_{k+1}]P^n_{k-1/2}(\lambda/2) \quad (k \geqslant 1),$$

$$Z^j_0 = (-(2n+1)d^j_1 + 8n\alpha_{nj})P^n_{-1/2}(\lambda/2).$$

In the calculation of $\sigma_{zn} + \sigma_{rn} + \sigma_{\theta n}$, the coefficients Z_k are set equal to Z^1_k.

The coefficients T_k of the series (15) have the form

$$T_k = l_k + q_k/2, \tag{16}$$

where

$$L_k = [(d^1_k + d^2_k + d^3_k) - (d^1_{k+1} + d^2_{k+1} + d^3_{k+1})]P^{n+1}_{k-1/2}(\lambda/2) \quad (k \geqslant 1),$$

$$L_0 = -(d^1_1 + d^2_1 + d^3_1)P^{n+1}_{-1/2}(\lambda/2),$$

$$q_k = E_{k-1} - E_{k+1} \quad (k \geqslant 2), \qquad q_1 = \binom{2}{0} E_0 - E_1, \qquad q_0 = -E_1,$$

$$E_k = (h_k - h_{k+1})P^{n+1}_{k-1/2}(\lambda/2) \quad (k \geqslant 1), \qquad E_0 = -h_1 P^{n+1}_{-1/2}(\lambda/2).$$

The formulas for U_k are similar to (16) for

$$L_k = [(d^1_k + d^2_k - d^3_k) - (d^1_{k+1} + d^2_{k+1} - d^3_{k+1})]P^{n+1}_{k-1/2}(\lambda/2) \quad (k \geqslant 1), \qquad L_0 = -(d^1_1 + d^2_1 - d^3_1)P^{n+1}_{-1/2}(\lambda/2).$$

2. On the cavity surface, we specify external forces p_z, p_r, and p_θ which are expanded in trigonometric series of the form (4). The coefficients of these series are related to the stress components by the formulas

$$p_{zn} = \sigma_{zn}\cos\alpha + \tau_{zrn}\sin\alpha, \qquad p_{rn} = \tau_{zrn}\cos\alpha + \sigma_{rn}\sin\alpha, \qquad p_{\theta n} = \tau_{z\theta n}\cos\alpha + \tau_{r\theta n}\sin\alpha, \tag{17}$$

where α is the angle between the outward normal to the surface and the axis Oz:

$$\sin\alpha = -\frac{\cosh\xi_0\cos\eta - 1}{\cosh\xi_0 - \cos\eta}, \qquad \cos\alpha = -\frac{\sinh\xi_0\sin\eta}{\cosh\xi_0 - \cos\eta}.$$

Let the specified external forces be also expanded in trigonometric series in the coordinate η of the toroidal coordinate system:

$$p_{zn} = \frac{1}{4c}(\lambda - 2\cos\eta)^{-3/2}\sum_{k=0}^{\infty}(P^s_{zk}\sin k\eta + P^c_{zk}\cos k\eta),$$

$$p_{rn} = \frac{1}{4c}(\lambda - 2\cos\eta)^{-3/2}\sum_{k=0}^{\infty}(P^s_{rk}\cos k\eta - P^c_{rk}\sin k\eta), \tag{18}$$

$$p_{\theta n} = \frac{1}{4c}(\lambda - 2\cos\eta)^{-3/2}\sum_{k=0}^{\infty}(P^s_{\theta k}\sin k\eta + P^c_{\theta k}\cos k\eta).$$

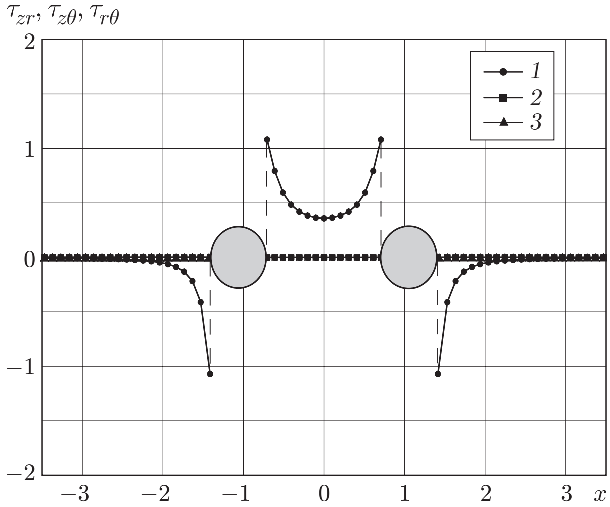

Fig. 3

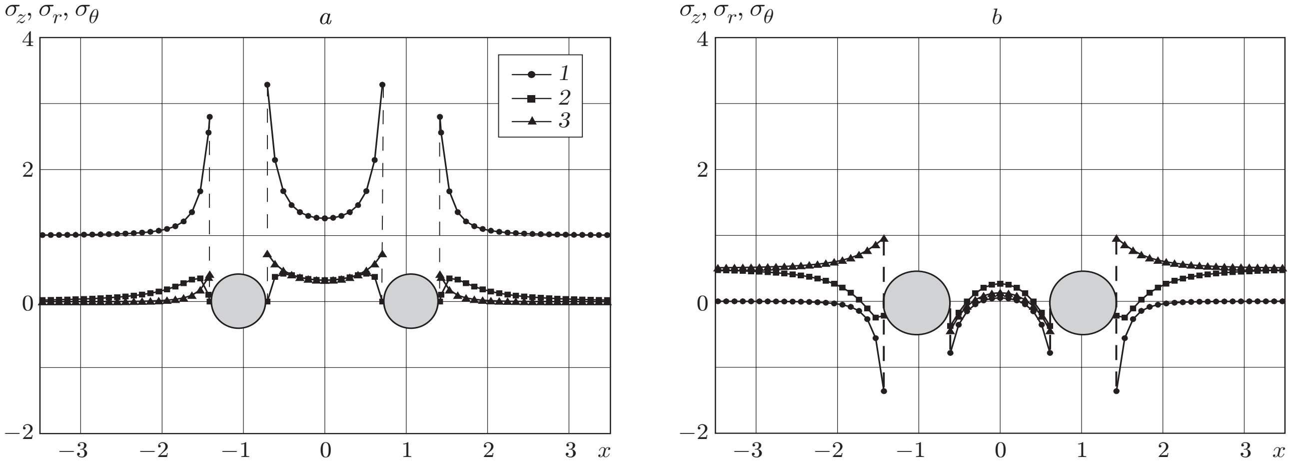

Fig. 4

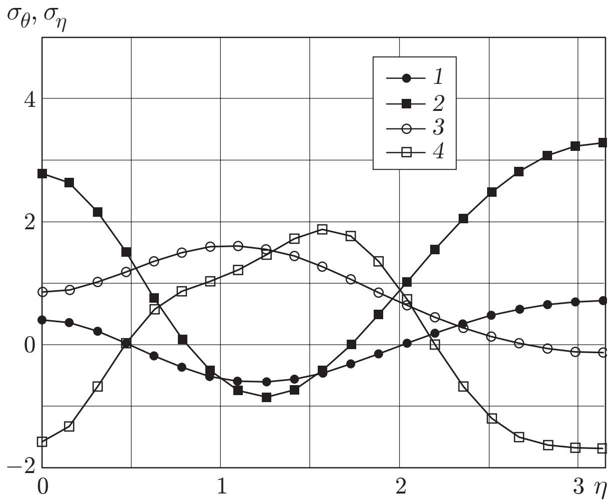

Fig. 5

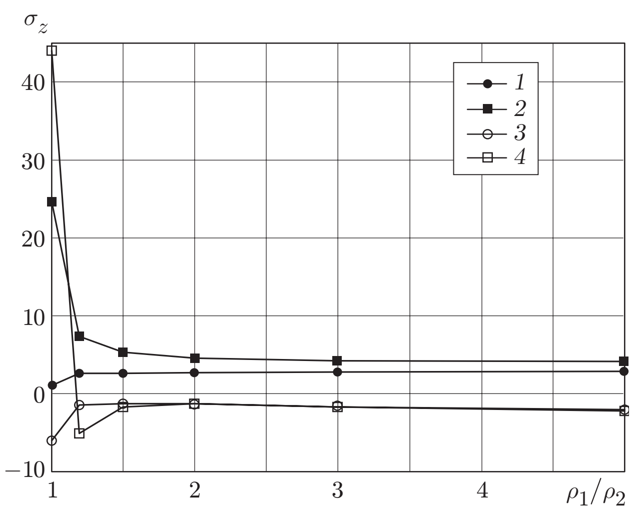

Fig. 6

Substituting (18) into equalities (17) and taking into account the series (15) we obtain

$$P_{zk} = \sqrt{(\lambda/2)^2 - 1}\,(S_{k+1} - S_{k-1}) + \lambda(T_{k+1} + U_{k+1} + T_{k-1} + U_{k-1})/2 - 2(T_k + U_k),$$

$$P_{rk} = \sqrt{(\lambda/2)^2 - 1}\,(T_{k+1} + U_{k+1} - T_{k-1} - U_{k-1})$$

$$+ \lambda(S_{k+1} - Z_{k+1} + S_{k-1} - Z_{k-1})/2 - 2(S_k - Z_k),$$

$$P_{\theta k} = \sqrt{(\lambda/2)^2 - 1}\,(T_{k-1} - U_{k-1} - T_{k+1} + U_{k+1}) - \lambda(V_{k-1} + V_{k+1})/2 + 2V_k \quad (k \geqslant 2),$$

$$P_{z1} = \sqrt{\Big(\frac{\lambda}{2}\Big)^2 - 1}\,\Big[S_2 - \binom{2}{0}S_0\Big] + \frac{\lambda}{2}\Big[T_2 + U_2 + \binom{0}{2}(T_0 + U_0)\Big] - 2(T_1 + U_1),$$

$$P_{r1} = \sqrt{\Big(\frac{\lambda}{2}\Big)^2 - 1}\,\Big[T_2 + U_2 - \binom{0}{2}(T_0 - U_0)\Big] + \frac{\lambda}{2}\Big[S_2 - Z_2 + \binom{2}{0}(S_0 + Z_0)\Big] - 2(S_1 - Z_1), \qquad (19)$$

$$P_{\theta 1} = \sqrt{\Big(\frac{\lambda}{2}\Big)^2 - 1}\Big[\binom{0}{2}(T_0 - U_0) - T_2 + U_2\Big] - \frac{\lambda}{2}\Big(V_2 + \binom{2}{0}V_0\Big) + 2V_1,$$

$$P_{z0} = \sqrt{(\lambda/2)^2 - 1}\,S_1 + \lambda(T_1 + U_1)/2 - 2(T_0 + U_0),$$

$$P_{r0} = \sqrt{(\lambda/2)^2 - 1}\,(T_1 + U_1) + \lambda(S_1 - Z_1)/2 - 2(S_0 - Z_0),$$

$$P_{\theta 0} = \sqrt{(\lambda/2)^2 - 1}\,(-T_1 + U_1) - \lambda V_1/2 + 2V_0.$$

Substituting the expressions of the quantities in terms of the coefficients of the series (7) into the right side of (19) and taking into account (6), we obtain an infinite system of equations.

6. Examples of Solutions of Particular Problems for an Elastic Space with a Toroidal-Shaped Cavity. Below we give results of solution of the following problems: rotation of a rigid torus in an elastic space around the axis Oy through an angle $\omega = 1$ (problem A); stretching of an elastic space by forces $\sigma_z^0 = 1$ applied at infinity (problem B); unilateral stretching by a stress $\sigma_x^0 = 1$ at infinity (problem C).

Figure 3 gives the distributions of tangential stresses along the line $z = 0$, $\theta = 0$ for problem A (curve 1 refers to τ_{zr}, curve 2 to $\tau_{z\theta}$, and curve 3 to $\tau_{r\theta}$). Figure 4a and b gives curves of tangential and normal stresses along the line $z = 0$, $\theta = 0$ for problems B and C, respectively (curve 1 refers to σ_z, curve 2 to σ_r, and curve 3 to σ_θ). In all cases, stress concentration is observed at points on the cavity surface at which $z = 0$. Figure 5 shows the stress variation from the point N to M (see Fig. 1) on the cavity surface in the section $\theta = 0$ at $\xi = \xi_0$ for

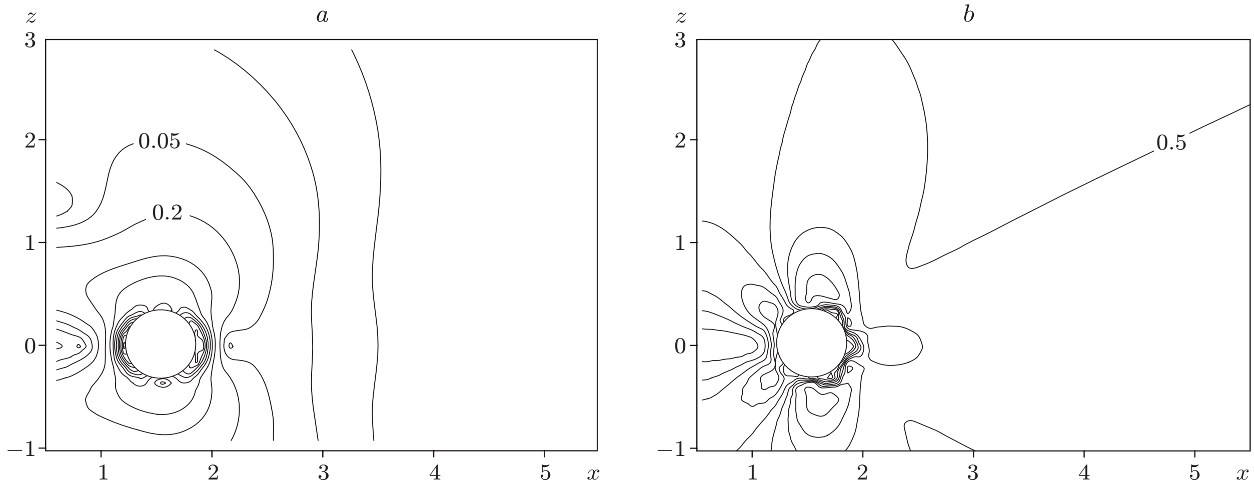

Fig. 7

problem B (curves 1 and 2) and problem C (curves 3 and 4). Here curves 1 and 3 correspond to σ_θ and curves 2 and 4 to σ_η.

The extreme values of the stresses are as follows: $\sigma_\theta = -0.5995$ for $\alpha = 3.1145$ and $\sigma_\eta = -0.8335$ for $\alpha = 3.1145$ in problem B and $\sigma_\theta = -1.613$ for $\alpha = -0.143$ and $\sigma_\eta = 1.884$ for $\alpha = 2.802$ in problem C.

Curves of the stresses σ_z at the points N and M versus the ratio of the radii ρ_1/ρ_2 are presented in Fig. 6 (curves 1 and 3 correspond to $\eta = 0$ and curves 2 and 4 to $\eta = \pi$) for problem B (curves 1 and 2) and problem C (curves 3 and 4). In the case $\rho_1/\rho_2 \approx 1$ (the inner radius of the torus cross section tends to zero) a sharp increase in the stresses is observed.

The isolines of the maximum tangential stresses for problems B and C are given in Fig. 7a and b, respectively.

Conclusions. The results obtained allow one to refine calculations of the stress–strain state of different objects, for example, internal defects such as ring cracks or mine openings around pillars at great depth and to improve the reliability and profitability of such calculations.

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